Assignment 4.

Polynomials, exponential, related functions.

This assignment is due Wednesday, Feb 13. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) (a) Prove that $z = z_0$ is a zero of multiplicity k of the polynomial $f(z)$ if and only if

$$f(z_0) = f'(z_0) = \ldots = f^{(k-1)}(z_0) = 0, \quad f^{(k)} \neq 0.$$

(*Hint:* It was proved in class that a zero $z = z_0$ of $f(z)$ has multiplicity k iff it is of multiplicity $k-1$ as a zero of $f'(z)$. Use this.)

(b) Prove that if $z_0 = a + bi$ is a zero of multiplicity k of the polynomial $f(z)$ with real coefficients, then $\overline{z_0} = a - bi$ is also a zero of multiplicity k of $f(z)$. (*Hint:* Conjugate the formula in the item 1a above.)

(c) Prove that every polynomial with real coefficients decomposes as a product of linear and quadratic polynomials with real coefficients. (*Hint:* Look at $(z - z_0)(z - \overline{z_0})$.)
Comment. This is one of many examples where the fact purely about real numbers is proved using complex numbers.

(d) Decompose x^4+1 in a product of two polynomials with real coefficients.

(2) Give another proof of existence and uniqueness of the exponential function (Theorem 10 in Sec. 4.2 of lecture notes), following the steps below:

(a) Prove that $f'(0) = 1$. (*Hint:* Argue why the limit $\lim\limits_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$ must be the same as $\lim\limits_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$, $x \in \mathbb{R}$.)

(b) Prove that $f'(z) = f(z)$ for all $z \in \mathbb{C}$. (*Hint:* Use the limit of ratios definition of derivative and the result above.)

(c) Treating the resulting differential equation $f' = f$, $f(0) = f'(0) = 1$ as a system of real-number differential equations, prove that the only complex differentiable function f that satisfies it is $f(z) = e^x(\cos y + i \sin y)$. (*Hint:* For example, if $f = u + iv$, you get $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, so $u = \frac{\partial u}{\partial x}$ and $v = \frac{\partial v}{\partial x}$. Don't forget Cauchy–Riemann equations.)

(3) It is known from calculus that $\lim\limits_{n\to\infty}(1 + \frac{x}{n})^n = e^x$ $(x \in \mathbb{R})$. Follow the steps below to prove that $\lim\limits_{n\to\infty}(1 + \frac{z}{n})^n = e^z$ for any $z \in \mathbb{C}$.

(a) Fix a complex number z. Denote $z_n = (1 + \frac{z}{n})^n$. Find $\lim\limits_{n\to\infty} |z_n|$.

Hint: Note that $|z_n| = |1 + \frac{x}{n} + i\frac{y}{n}|^n == \left((1 + \frac{x}{n})^2 + \frac{y^2}{n^2}\right)^{n/2}$. Argue that you can discard terms with $1/n^2$; for example, by showing that the limit $\lim\limits_{n\to\infty} \frac{\left((1+\frac{x}{n})^2 + \frac{y^2}{n^2}\right)^{n/2}}{\left(1+\frac{2x}{n}\right)^{n/2}}$ is equal to 1.

(b) Find $\lim\limits_{n\to\infty} \operatorname{Arg} z_n$. (In particular, pick appropriate values of Arg for each n so that the limit exists.)
Hint: Show that $n \arctan \frac{y/n}{1+(x/n)} \in \operatorname{Arg} z_n$. Argue that you can replace arctan with its argument as $n \to \infty$; for example, by showing that $\lim\limits_{n\to\infty} \frac{\arctan \frac{y/n}{1+(x/n)}}{\frac{y/n}{1+(x/n)}} = 1$.

(c) Combine results of (a) and (b) to get $\lim\limits_{n\to\infty} z_n$.

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(4) (a) (*Logarithmic spiral.*) Find and sketch the image of a straight line

$$z = (1 + i\alpha)t + ib, \quad -\infty < t < +\infty,$$

$\alpha, b \in \mathbb{R}$, $\alpha \neq 0$, under the map $w = e^z$. Eliminate t in the answer to get an equation in polar coordinates. (This straight line has slope α (meaning it intersects x-axis at the angle $\arctan \alpha$) and intersects y-axis at the point ib.)

(*Hint:* After eliminating t, the answer should be $r = ce^{\varphi/\alpha}$, where $c = e^{-b/\alpha}$. This curve is called a *logarithmic spiral.*)

(b) Use conformity of $w = e^z$ to find the angle at which logarithmic spiral intersects rays emanating from the origin.

(c) Given $\alpha \in \mathbb{R}$, find the condition on real numbers $c, c' > 0$ for the spirals $r = ce^{\varphi/\alpha}$ and $r = c'e^{\varphi/\alpha}$ to be the same (as sets on a plane).
Comment. In other words, logarithmic spiral is *self-similar.*

(d) Find and sketch the image under $w = e^z$ of the slanted strip S with sides

$$y = k(x - a_1), \quad y = k(x - a_2) \qquad (k, a_1, a_2 \in \mathbb{R};\ k \neq 0).$$

(*Hint:* Convert equations of straight lines to the form used in item 4a.)

(e) Find the condition on k, a_1, a_2 for the map $\exp : S \to \mathbb{C}$ to be injective, (i) using item 4c above, (ii) using periodicity of exp.

(5) (a) Find $(\cos z)', (\sin z)', (\sinh z)', (\cosh z)'$. (Probably, the quickest way is to express these functions through exp.)

(b) Find all complex solutions of equations $\cos z = 0$, $\sin z = 0$, $\cosh z = 0$, $\sinh z = 0$.

(c) Using formulas for $\cos(z_1 + z_2)$ and $\sin(z_1 + z_2)$, express $|\cos z| = |\cos(x + iy)|$ and $|\sin z| = |\sin(x + iy)|$ through $\sin, \cos, \sinh, \cosh$ of x and y.